COMMUNICATING MATHEMATICS: HOW LANGUAGE FORMS MAKE IN/ACCESSIBLE MATHEMATICALLY IN/APPROPRIATE CALCULUS CONCEPTUALIZATIONS

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The study reported here was designed to investigate student learning in calculus with a focus on language use and the ways truth and validity are determined. Results reported here are those related to students' processes of construction of particular mathematics conceptualizations as a result of exposure to three different approaches to calculus instruction: technique-oriented, concepts-first and infinitesimal instruction. When students used infinitesimal language and used it in conjunction with everyday language they generally did so as a foundation by which to construct mathematically valid problem responses. This finding indicates that instruction emphasising connections between everyday and technical language is likely to guide students to build mathematically appropriate inter-connected conceptualizations. Also, the use of infinite magnification in a variety of problem situations by students who received infinitesimal instruction demonstrates that instruction emphasising visual interpretations can influence students' conceptualizations.

A whole body of research in mathematics education in the last decade has focussed on students' interpretations of mathematical concepts and processes. This has included documenting how "students bring pre-mathematical experiences into the classroom which affect their understandings of the mathematics" (Tall, 1990; p.49). Researchers have found that students hold "mini-theories" about mathematical ideas and that they learn mathematics in ways that are "personally reasonable and sensible" (Confrey, 1992; p.122). That is, students' mathematical models, though not necessarily in congruence with those of a teacher or researcher, are reasonable to themselves. As alternative perspectives these models are viable and legitimate within a certain range of situations and applications. (For reviews of this literature, see Confrey, 1990; Perkins and Simmons, 1987; Driver and Easley, 1978).

Whether calling students' mathematical models personal conceptions, mini-theories, alternative theories, inadequate beliefs or misconceptions, it is evident that "the theories and their forms of argument must be addressed if students are to come to a more acceptable understanding of the concept" (Confrey, 1992; p.121). Consequently, as teachers and researchers attempt to develop learning experiences that will help students construct mathematically appropriate conceptualizations, they must be informed as to how mathematics might be effectively communicated. The inherent ambiguities of communication must be considered, particularly in relation to the symbolic and verbal language forms that are prominent elements of communication within mathematics classrooms. Thus, there is a need for mathematics educators to consider how students' constructive processes and related conceptualizations are mediated by various mathematical symbols, technical and everyday language.

In relation to students' mathematical representations, an area of mathematics education in which researchers have identified a persistent and recurring phenomena is that of calculus learners' "misconceptions" of calculus concepts. The term "misconception" is used here and in the upcoming discussions, rather than alternative conception, personal conception or some other phrase, because it is the term employed most commonly in the research literature. Examples of student misconceptions in calculus have been noted with learners on several continents and across a range of topics. In particular, investigations with calculus students have documented misconceptions present in their understandings of limits, infinity, continuity, tangents, derivatives and integration (for example, Artigue, 1986; Cornu, 1981; Davis and Vinner, 1986; Orton, 1977, 1983a, 1983b; Schwarzenberger and Tall, 1978; Sierpinska, 1987; Tall, 1989; Tall and Vinner, 1981; Williams, 1991). These investigations of student learning in calculus have given insight into students' misconceptions. What is needed now is research into how instruction can better guide and support student learning in calculus. The results of the study reported here have practical implications in relation to this last point.

DESCRIPTION OF THE STUDY

Research Setting

The research was a naturalistic study involving three undergraduate calculus classes located at three different post-secondary institutions in Western Canada. These institutions included a large university and two small private colleges. The course at the university was representative of introductory calculus courses in its content and an emphasis upon learning techniques for differentiation, integration, graphing and problem solving. In comparison, one of the colleges used a "concepts-first" approach to instruction in which concepts are explored intuitively before introduction of their formal definitions and proofs and before skill development is emphasised. The second college used an instructional approach which develops concepts intuitively while using infinitesimal methods related to nonstandard analysis as analytic and computational tools. Infinitesimal methods are the tools by which Newton and Leibniz first developed calculus in the late 1600's.

The best way to demonstrate how an infinitesimal approach to instruction differs from the use of methods in real analysis and in particular from technique-oriented and concepts-first instruction is to provide some specific examples of its use. Two appropriate examples are the following:

(1) Limits and their precise $\varepsilon-\delta$ definition are replaced by the more intuitive notion of "rounding off", denoted by $\rightsquigarrow$ (an idea students have used since elementary school). (See Robinson (1966) for a complete and mathematically rigorous account of the development of calculus using infinitesimal numbers, called nonstandard analysis).

(2) The derivative is not introduced via rotating secants which in the limit become a tangent line at a point on a graph. Rather, the value of the derivative at a point is the slope of the tangent line at that point (if the tangent line exists). This concept of derivative is introduced after tangent lines (and where they do and do not exist) have been introduced via the intuitive notion of magnification.

Research Methods

Task-based interviews with 17 students were the primary method of inquiry into the nature and role of students' language use. These in depth interviews involved students in oral and written responses to a number of calculus problems focusing on calculus skills and concept interpretations. Students' language use was initially examined on the broad level of an entire problem response. This examination was done by counting occurrences of a student's use of symbols and technical or everyday language terminology that were not given in the problem statement. In relation to symbol use, manipulations or operations with symbols present in the statement of a problem were distinguished from a student's use of a symbolic representation not present in the problem statement. Results of the counts of students' use of symbols, technical and everyday language terms were used to determine the nature of their language use, while the role of their language use was determined from more extensive examination of what they said or wrote and what this language reflected of their calculus conceptualizations. It is this latter extensive examination that is reported in this paper.

The range of instructional settings allowed partial examination of the impact of different approaches to instruction on the nature and role of students' language use, although the small number of students interviewed at each institution did not permit statistical analysis or definitive answers on the effect of instruction upon students' calculus learning. The examination of students' problem responses did however give insight into the potential impact of each instructional approach on all students' learning.

RESULTS AND RELATED DISCUSSION

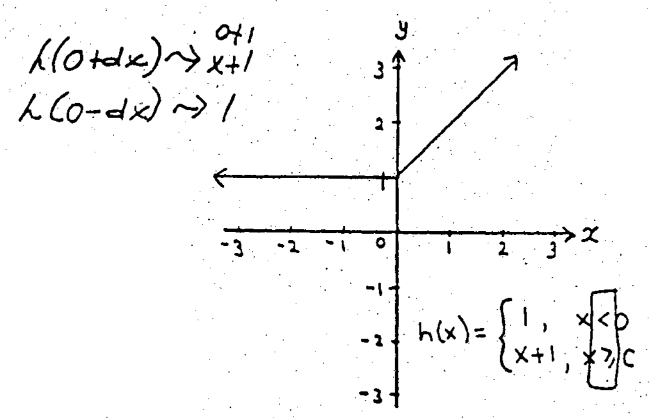
Students' problem responses revealed some important features of their language use, including: (1) conceptualizations built using infinitesimal language displayed features different from conceptualizations built using traditional calculus language, and (2) whether speaking with traditional or infinitesimal language, students used terminology as tools by which to conceptualise their descriptions and explanations and in these constructions pre-calculus language knowledge was prominent. In relation to point (1) it must be noted that students who received infinitesimal instruction, including those students who had studied some calculus previously using traditional language and symbols, used symbols and words particular to infinitesimal calculus. In particular, it will be seen as this discussion proceeds that infinitesimal symbols served students as objects that could be concretely represented on a graph and referred to and used as tools for construction of an explanation or justification.

Although symbols did not form a large component of students' language use, with more than half the students using symbolic representations in one third or less of their problem responses, students who received infinitesimal instruction made more use of symbols. Unlike most of the other students, most of the students who received infinitesimal instruction were able to give symbolic justifications or explanations of continuity and differentiability. Furthermore, the symbols they used and their corresponding verbal language (both everyday and techincal) were particular to an infinitesimal approach to instruction. An exampleof one of these responses is the following:

(Tanya)

(Problem 5)

[5. For each function given below, determine if it is continuous or discontinuous. Give reasons for your answer.]



That any, I'll kind of do it this way. y at x. And these two x's are the same. Ah. If you take any x point and go a little bit to the left or a little bit to the right an infinitesimal amount, it rounds off to y at that x on the y-axis.

Figure 1. Tanya's Response to Problem 5

In this extract Tanya uses infinitesimal language (words and symbols both) to explain the relationship between the behaviour of a graph and the corresponding notion of continuity. In doing so, infinitesimal notation serves as a primary tool for construction of an explanation. It is a key tool in that interpretation of dx as an infinitesimal number provides Tanya with something fairly concrete to work with. She easily visually locates on a graph what dx corresponds to and how the position of dx relates to the behaviour of the graph. The role of her use of infinitesimal notation is both to build and to justify her response.

In relation to technical language use, students who received infinitesimal instruction used technical language to about the same degree as students at the other two institutions, but they used everyday language more. Although this finding distinguishes them from the other students, what distinguishes them more is the content of their technical and everyday language use and the role of this language in describing, explaining or justifying calculus ideas. These features will now be discussed, pointing out the nature and role of language in students' interpretations of calculus problems.

Students who received infinitesimal instruction generally integrated everyday language more with symbols or technical language than did the other students. In comparison, students who received technique-oriented or concepts-first instruction, although they often gave valid explanations of situations using everyday language, did not as frequently use technical terms or symbols for further, more detailed or precise justifications. In particular, unless specifically asked to do so, they did not make use of language and ideas related to limits. There were also occasions when they used technical terms or symbols but were unable to explain their connections to everyday language explanations.

Another aspect of infinitesimal instruction which was displayed in students' problem responses and which appeared as important in these responses was the notion of magnifying a curve. At some point in their interviews all students who received infinitesimal instruction spoke of infinitely "magnifying" or "blowing up" the graph of a function. In an infinitesimal approach to calculus instruction magnification is a means by which a function can be examined "up close". In this process infinitesimal language plays a role in students' interpretations by orienting them to construct descriptions of a magnified curve. Non-infinitesimal language related to the slope of a tangent line also served to orient students to descriptions of a curve. However, these descriptions, justifications and conclusions seldom made use of limit-related language or processes. In comparison, the notion of infinite magnification has limiting processes built into its use. This feature distinguishes it from traditional slope and tangent line notions in more than one way. First, it is a dynamic rather than static method for interpretation of graphs. Second, magnification makes the limit concept of "close to" accessible. That is, the visual mechanism of blowing up or infinitely magnifying a curve serves as a visual, physically accessible means by which to examine related limiting notions.

The traditional limit concept also has visual interpretations, but these were not regularly used by students who received technique-oriented or concepts-first instruction. In fact, the general absence of use of limit notation or terminology by students who received these approaches to instruction, unless it was specifically requested, indicates they did not integrate their limit conceptualizations into other calculus conceptualizations. For example, their responses included explanation of the derivative as the limit of slopes of a sequence of secant lines, but the relationship of limits and derivatives was then not applied in other problem responses. Use of the notion of magnification was more regularly applied by students as a tool by which to construct calculus conceptualizations.

In particular, this study found differences in the nature of student problem responses that related to whether or not they made use of infinitesimal numbers or the notion of infinite magnification. A feature of the problem responses of students who received infinitesimal instruction was that when they used magnification and related terminology they did not construct the same misconceptions present in problem responses of students who did not use infinitesimal terminology (including incidents when students who received infinitesimal instruction did not use infinitesimal terminology). For example, students who did not use infinitesimal language tended to interpret the technical language term "continuous" using everyday language phrases such as "no breaks", "no jumps", "existing", "being defined" and "not changing". Many of their notions associated with these everyday language phrases were valid interpretations of situations, although they were not necessarily valid mathematical interpretations. The interpretations therefore tended to guide students to construct mathematically incorrect justifications or justifications that were used inconsistently. For example, interpretation of the technical language term "continuous" in terms of "existing" led students to construct mathematically incorrect justifications. Further, although interpretation of "continuous" as "no breaks" usually oriented students to mathematically correct notions related to continuity, it did not do so for all of them. For example, Doug believed that a "break" in the way a function is defined constitutes a discontinuity.

Another misconception displayed by students who did not use the notion of magnification was that non-uniqueness rather than non-existence of a tangent line implies non-differentiability. For example, students said such things as:

(Cindy)
(Problem 9)
[9. The graph of $y = F(x)$ is given below. At which points does the function not have a derivative? Why?]
Because this is undefined [at a cusp]. Because a derivative means you're taking the slope of a tangent. But the tangent, it could be here, it could be here, it could be here. It could be anywhere. And we don't know where it is.
(Annabel)
(Problem 9)
Derivative is suppose to ah, on a graph the derivative is supposed to be a tangent line that touches the graph at only one spot. And at a sharp point or an endpoint there it touches it, it can do that in many different places. So you cannot define any one derivative.

The students to whom the above interview excerpts belong gave the correct conclusion that no derivative existed at a particular point, but their justifications for nondifferentiability were mathematically incorrect. In comparison, students who employed the notion of magnification to determine differentiability said such things as:

(Gordon)
(Problem 9)
The line has to be continuous. So you wouldn't have one [a tangent] at the endpoint. [pause] If you blow that up, infinitesimally you still have that. You can't draw a tangent to that. Then you can't have a derivative.
(Tanya)
(Problem 9)
Right at this point if you magnify it. You're magnifying the point and you still have a straight line. In order to have a derivative you need a line. You don't need a point and a line to the left or right of it. You need a line where you can draw a tangent line and a slope to it. Here, like I said, a derivative just to the right of it exists [at a point of discontinuity]. Left, sorry. Just to the left it exists. Infinitesimally. Right at that point it doesn't exist.
(Nadine)
(Problem 9)
. . . you take the point and you blow it up an infinite amount. And if you see a straight line there's a derivative. . . . You'll still see this. You blow it up and you'll still see a V [at a cusp]. And at this point there is no derivative.

The above excerpts show how magnification of a curve generally served to focus students' perceptions and subsequent justifications upon non-existence rather than non-uniqueness of a tangent line.

In summary, students who received infinitesimal instruction used symbols and words particular to infinitesimal calculus as primary tools to explain or justify calculus ideas. They used infinitesimal language and related visual notions such as infinitesimal closeness and infinite magnification so that infinitesimal symbols served them as objects that could be concretely represented on a graph and referred to in construction of an explanation or justification. Students who used infinitesimal language, more frequently than the other students, appropriately integrated technical and everyday language and did not display the same misconceptions present in other students' responses. All students often gave valid physical interpretations for visually oriented notions such as continuity, slope or size, but students who used infinitesimal language had both the tools and mechanisms by which to make different constructions that did not lead to the same misconceptions. In this way, since students who used infinitesimal language did so as a means to regulate their actions, infinitesimal language made accessible mathematically appropriate thought.

IMPLICATIONS

The implications for calculus instruction of this study's findings are twofold:

(1) When students used infinitesimal language and used it in conjunction with everyday language they generally did so as a foundation by which to construct mathematically valid problem responses. This finding indicates that instruction emphasising connections between everyday and technical language is likely to guide students to build mathematically appropriate inter-connected conceptualizations.

(2) The use of infinite magnification in a variety of problem situations by students who received infinitesimal instruction demonstrates that instruction emphasising visual interpretations can influence students' conceptualizations. It can guide students to use physical, bodily experiences as a means by which to articulate abstract ideas and construct related conceptualizations. Most importantly, construction of calculus meanings from the physical experiences of magnification and closeness did not make accessible to students the misconceptions present in the conceptualizations of students who had received technique-oriented or concepts-first instruction. That is, use of infinitesimal language did not facilitate expression of mathematically inappropriate or inaccurate conceptualizations.

The implications of these points for mathematics education at all levels, not just for the teaching and learning of calculus, are both relevant and useful. Practical use can be made of the notion that students can be guided to build mathematically appropriate inter-connected conceptualizations by instruction emphasising connections between everyday and technical language. Specifically, the design and implementation of instruction should provide students with opportunities to construct conceptualizations through elaboration of their everyday language meanings. Use of everyday language interpretations to describe, explain and justify particular mathematical situations could then be connected to more abstract symbolic mathematical representations.

More importantly, practical use can be made of the notion arising from this study that students will model mathematical concepts using language forms that reflect their instructional experiences. In particular, mathematics teaching at all levels should consider that some instructional language forms and related experiences are more conducive to facilitating students' construction of mathematically correct or appropriate thought, while rendering inaccessible certain misconceptions. That is, it is possible to design instructional experiences which are language based and which do not provide students with the vocabulary to frame certain misconceptions. The experiences and related vocabulary, by inhibiting certain misconceptions, facilitate construction of more mathematically appropriate conceptualizations.

REFERENCES

Artique, M. (1986). The notion of differential for undergraduate students in the sciences. In *Proceedings of the Tenth International Conference of Psychology of Mathematics Education,* University of London, London, 229-234.

Confrey, J. (1990). A review of research on student conceptions in mathematics, science and programming. In Cazden, C. (Ed.). *Review of Research in Education,* 16, American Education Research Association, Washington, D.C., 3-56.

Confrey, J. (1992). Learning to listen: a student's understanding of powers of ten. In von Glasersfeld, E. (Ed.). *Radical Constructivism in Mathematics Education.* Kluwer Academic Publishers, Dortrecht.

Cornu, B. (1981). Apprentissage de la notion de limite: modeles spontanes et modeles propres. In *Actes due Cinquieme Colloque du Groupe International P.M.E.,* Grenoble, 322-326.

Davis, R. & Vinner, S. (1986). The notion of limit: some seemingly unavoidable misconception stages. *Journal of Mathematical Behavior,* 5, 3, 281-303.

Driver, R. and Easley, J. (1978). Pupils and paradigms: a review of literature related to concept development in adolescent science students. *Studies in Science Education,* 5, 61-64.

Orton, A. (1977). Chords, secants, tangents and elementary calculus. *Mathematics Teaching,* 78, 48-49.

Orton, A. (1983a). Students' understanding of integration. *Educational Studies in Mathematics,* 14, 1, 1-18.

Orton, A. (1983b). Students' understanding of integration. *Educational Studies in Mathematics,* 14, 3, 235-250.

Perkins, D. and Simmons, R. (1987). Patterns of misunderstanding. An integrative model of misconceptions in science, math, and programming. In Novak, J. (Ed.). *Proceedings of the Second International Seminar on Misconceptions and Educational Strategies in Science and Mathematics,* Ithaca, 381-395.

Schwarzenberger, R. and Tall, D. (1978). Conflicts in the learning of real numbers and limits. *Mathematics Teaching,* 82, 44-49.

Sierpinska, A. (1987). Humanities students and epistemological obstacles related to limits. *Educational Studies in Mathematics,* 18, 371-397.

Tall, D. (1989). Concept images, generic organizers, computers, and curriculum change. *For the Learning of Mathematics,* 9, 3, 37-42.

Tall, D. (1990). Inconsistencies in the learning of calculus and analysis. *Focus on Learning Problems in Mathematics,* 12, 49-63.

Tall, D. & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics,* 12, 2, 151-169.

Williams, S.R. (1991). Models of limit held by college calculus students. *Journal for Research in Mathematics Education,* 22, 3, 219-236.